Exit time assymptotics on non-commutative 2-torus.

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The purpose of this talk is to establish an analogue of exit time asymptotics of Brownian motion on manifolds, in the set-up of non-commutative 2-torus. Using these asymptotics, we will try to formulate definitions of certain geometric invariants e.g. intrinsic dimension, mean curvature etc for the non-commutative 2-torus.

Outline of the talk

1. Interplay between Geometry and Probability:
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   - Exit time asymptotics of Brownian motion on manifolds.
2. Formulation of quantum exit time.

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We begin with the following well-known proposition:

Pinsky,1994

Consider a hypersurface $M \subseteq \mathbb{R}^d$ with the Brownian motion process X_t^m starting at m. Let $T_\varepsilon = inf\{t > 0 : \|X_t^m - m\| = \varepsilon\}$ be the exit time of the motion from an extrinsic ball of radius ε around m. Then we have

$$\mathbb{E}_m(T_\varepsilon) = \varepsilon^2/2(d-1) + \varepsilon^4 H^2/8(d+1) + O(\varepsilon^5),$$

where H is the mean curvature of M.

Gray,1973

Let $V_m(\epsilon)$ denote the volume of a ball of radius ϵ around $m \in M$. Let n be the intrinsic dimension of the manifold. Then we have

$$V_m(\epsilon) = \frac{\alpha_n \epsilon^n}{n} \left(1 - K_1\epsilon^2 + K_2\epsilon^4 + O(\epsilon^6)\right)_m ,$$

where $\alpha_n := 2\Gamma(\frac{1}{2})^n \Gamma(\frac{n}{2})^{-1}$ and K_1, K_2 are constants depending on the manifold.

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The intrinsic dimension n of the hypersurface M is the unique integer n satisfying $\lim_{\epsilon\to 0} \frac{\mathbb{E}(\tau_\epsilon)}{V_\epsilon^{\frac{2}{m}}} = \begin{cases} \infty \text{ if } m \text{ is less than } n; \\ \neq 0 \text{ if } m \neq n; \\ = 0 \text{ if } m > n. \end{cases}$

Observe that $\frac{V(\epsilon)^{\frac{2}{n}}}{\epsilon^2} \longrightarrow \left(\frac{\alpha_n}{n}\right)^{\frac{2}{n}}$ and $\frac{V(\epsilon)^{\frac{4}{n}}}{\epsilon^4} \longrightarrow \left(\frac{\alpha_n}{n}\right)^{\frac{4}{n}}$ as $\epsilon \longrightarrow 0^+$.

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In view of this, the asymptotic expression appearing in Pinsky's result can be recast as

$$\mathbb{E}(\tau_\epsilon) = \frac{1}{2(d-1)}(\frac{V(\epsilon)n}{\alpha_n})^{\frac{2}{n}} + \frac{H^2}{8(d+1)}(\frac{V(\epsilon)n}{\alpha_n})^{\frac{4}{n}} + O(V(\epsilon)^{\frac{5}{n}}).$$

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$$d = \frac{1}{2}(1 + \lim_{\epsilon \longrightarrow 0} \frac{1}{\mathbb{E}(\tau_\epsilon)}(\frac{nV(\epsilon)}{\alpha_n})^{\frac{2}{n}}), \tag{1}$$

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$$H^2 = 8(d+1)(\frac{\alpha_n}{n})^{\frac{4}{n}} \lim_{\epsilon\longrightarrow 0} \frac{\mathbb{E}(\tau_\epsilon) - \frac{1}{2(d-1)}(\frac{nV(\epsilon)}{\alpha_n})^{\frac{2}{n}}}{V(\epsilon)^{\frac{4}{n}}}. \tag{2}$$

Formulation of quantum exit time.

Suppose that M is a Riemannian manifold of Dimension n. Let B_r^x be a ball of radius r around $x \in M$. Choose a coordinate neighbourhood $(U_x; x_1, x_2, ...x_n)$ around x. Let W_t^x be a Brownian motion on M starting at x and $\tau_{B_r^x}$ be the exit time of the Brownian motion from the ball B_r^x.

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$$\chi_{\{\tau_{B_r^x} > t\}} = \bigwedge_{s \leq t} \left(\chi_{\{W_s^x \in B_r^x\}} \right),$$

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Note that

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since by the Wiener- Itô isomorphism, $L^2(\mathbb{P}) \cong \Gamma(L^2(\mathbb{R}_+, \mathbb{C}^n))$, where $\mathbb{P}$ is the n dimensional Wiener measure.

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So one may write

$$\chi_{\{\tau_{B_r^x} > t\}}(\cdot) = \bigwedge_{s \le t} j_s(\chi_{B_r^x})(x, \cdot) = \bigwedge_{s \le t} ((ev_x \otimes id) \circ j_s(\chi_{B_r^x}))(\cdot).$$

Thus we may view $\tau_{B_r^x}$ as a spectral family in $L^\infty(U_x) \otimes B(\Gamma(L^2(\mathbb{R}_+, \mathbb{C}^n)))$ by the prescription:

$$\tau_{B_r^x}([0,t)) = \mathbf{1} - \wedge_{s \le t}(j_s(\chi_{B_r^x}))\ .$$

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Moreover, we have:

$$\mathbb{E}(\tau_{B_r^x}) = \int_0^\infty \mathbb{P}(\tau_{B_r^x} > t)dt = \int_0^\infty \langle e(0), \{(ev_x \otimes 1)\left(\wedge_{s \le t} j_s(\chi_{B_r^x})\right)\} e(0)\rangle dt.$$

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Alternatively:
Choose a sequence $(x_n)_n \in M$ and positive numbers ϵ_n such that $x_n \longrightarrow x$ and $\epsilon_n \longrightarrow 0$. Now for large n, $\chi_{\{W_s^{x_n} \in B_{\epsilon_n}^{x_n}\}}(\cdot) \stackrel{\mathcal{L}}{=} \chi_{\{W_s^x \in B_{\epsilon_n}^x\}}(\cdot)$ for each $s \geq 0$. Thus,

$$\mathbb{E}(\tau_{B_{\epsilon_n}^{x_n}}) = \int_0^\infty \langle e(0), \{(ev_{x_n} \otimes id)\left(\wedge_{s\leq t} j_s(\chi_{B_{\epsilon_n}^{x_n}})\right)\} e(0)\rangle dt = \mathbb{E}(\tau_{B_{\epsilon_n}^x}),$$

i.e. the asymptotic behaviour of $\mathbb{E}(\tau_{B_{\epsilon_n}^{x_n}})$ and $\mathbb{E}(\tau_{B_{\epsilon_n}^x})$ will be the same.

Note that the points of M are in $1-1$ correspondence with the pure states of $L^{\infty}(M)$ and $\{P_n = \chi_{B_{\epsilon_n}^{x_n}}\}_n$ is a family of projections on $L^{\infty}(M)$, so that we have:

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We now move into non-commutative setup.

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Barnett,Wilde, 1991

Let $(\mathfrak{A}_t)_{t\geq 0}$ be an increasing family of von-Neumann algebras (called a filtration). A quantum random time or stop time adapted to the filtration $(\mathfrak{A}_t)_{t\geq 0}$ is an increasing family of projections $(E_t)_{t\geq 0}$, $E_0 = I$ such that E_t is a projection in $\mathfrak{A}_t$ and $E_s \leq E_t$ whenever $0 \leq s \leq t < +\infty$.

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Observe that by our definition, $\tau_{B_r^x}([0,t))$ is adapted to the filtration $(\mathfrak{A}_t)_{t\geq 0}$, where
$\mathfrak{A}_t := L^\infty(U_x) \otimes B(\Gamma_{t]})$ $\left(\Gamma_{t]} := \Gamma\left(L^2([0,t],\mathbb{C}^n)\right)\right)$, for
$\tau_{B_r^x}([0,t]) \in \mathfrak{A}_t \otimes 1_{\Gamma_{[t}}$.

Suppose that we are given an E-H flow $j_t : \mathcal{A} \to \mathcal{A}'' \otimes B(\Gamma(L^2(\mathbb{R}_+, k_0)))$, where $\mathcal{A}$ is a C^* or von-Neumann algebra. For a projection $P \in \mathcal{A}$, the family $\{\mathbf{1} - \wedge_{s \leq t} (j_s(P))\}_{t \geq 0}$ defines a quantum random time adapted to the filtration $\left(\mathcal{A}'' \otimes B(\Gamma_{t]})\right)_{t \geq 0}$.

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Definition

We refer to the quantum random time $\{1 - \bigwedge_{s \leq t} j_s(P)\}_{t \geq 0}$ *as the 'exit time from the projection* P*.*

Let τ be a state (to be thought of as non-commutative volume form on a C^* or von Neumann algebra), and assume that we are given a family $\{P_n\}_{n\geq 1}$ of projections in $\mathcal{A}$, and a family $\{\omega_n\}_{n\geq 1}$ of pure states of $\mathcal{A}$ such that

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Definition

Let $\gamma_n := \int_0^\infty dt \langle e(0), (\omega_n \otimes id) \circ \bigwedge_{s\leq t} j_s(P_n) e(0)\rangle$. We say that there is an exit time asymptotic for the family $\{P_n; \omega_n\}$ of intrinsic dimension n_0 if

$$\lim_{n\to\infty} \frac{\gamma_n}{v_n^{\frac{2}{m}}} = \begin{cases} \infty \text{ if } m \text{ is just less than } n_0 \\ \neq 0 \text{ if } m \neq n \\ = 0 \text{ if } m > n \end{cases}$$

and

$$\gamma_n = c_1 v_n^{\frac{2}{n_0}} + c_2 v_n^{\frac{4}{n_0}} + \cdots c_k v_n^{\frac{2^k}{n_0}} + O(v_n^{\frac{2^{k+1}}{n_0}}) \text{ as } n \to \infty. \tag{3}$$

It is not at all clear whether such an asymptotic exists in general, and even if it exists, whether it is independent of the choice of the family $\{P_n; \omega_n\}$. If it is the case, one may legitimately think of $c_1, c_2, \ldots c_k \ldots$ as geometric invariants and imitating the classical formulae as discussed before, the extrinsic dimension d and the mean curvature H of the non-commutative manifold may be defined to be

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Definition

The non-commutative 2-torus $C^*(\mathbb{T}^2_\theta)$ is the universal C^*-algebra generated by a pair of unitaries U, V which satisfy:

$$UV = e^{2\pi i\theta} VU.$$

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It can also be viewed as the "Rieffel deformation" of the commutative C^*-algebra $C(\mathbb{T}^2)$.

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Choose an $\epsilon << \theta$ and let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$, where $f_1, f_0 \in C(\mathbb{T}^2)$, $f_{-1}(t) := \overline{f_1(t+\theta)}$,

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$$f_0(t) = \begin{cases} \epsilon^{-1}t \quad if\ 0 \leq t \leq \epsilon \\ 1 \quad if\ \epsilon \leq t \leq \theta \\ \epsilon^{-1}(\theta+\epsilon-t)\ if\ \theta \leq t \leq \theta+\epsilon \\ 0\ if\ \theta+\epsilon \leq t \leq 1 \end{cases}$$

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$$f_1(t) = \begin{cases} \sqrt{f_0(t) - f_0(t)^2} \ if \ \theta \leq t \leq \theta + \epsilon \\ 0 \ if \ otherwise. \end{cases}$$

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- Let tr be the canonical trace in $C^*(\mathbb{T}_\theta^2)$, given by $tr(\sum_{m,n} a_{mn} U^m V^n) = a_{00}$. This trace will be taken as an analogue of the volume form in $C^*(\mathbb{T}^2)$.

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- Throughout the section, we will assume $C^*(\mathbb{T}^2_\theta) \subseteq B(L^2(tr))$, and let $W^*(\mathbb{T}^2_\theta) := (C^*(\mathbb{T}^2_\theta))''$.

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- Throughout the section, we will assume $C^*(\mathbb{T}^2_\theta) \subseteq B(L^2(tr))$, and let $W^*(\mathbb{T}^2_\theta) := (C^*(\mathbb{T}^2_\theta))''$.
- For $(x, y) \in \mathbb{T}^2$, let $\alpha_{(x,y)}$ denote the canonical action of $\mathbb{T}^2$ on $C^*(\mathbb{T}^2_\theta)$ given by $\alpha_{(x,y)}(\sum_{m,n} a_{mn} U^m V^n) = \sum_{m,n} x^m y^n a_{mn} U^m V^n$. Note that the automorphism α is tr-preserving. Hence it extends to a unitary operator on $L^2(tr)$, say $u_{(x,y)}$, and $\alpha = ad\ u$, which implies that α is normal.

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- On $C^*(\mathbb{T}^2_\theta)$, there are two conditional expectations denoted by ϕ_1, ϕ_2, which are defined as:

$$\phi_1(A) := \int_0^1 \alpha_{(1, e^{2\pi it})}(A)dt, \quad \phi_2(A) := \int_0^1 \alpha_{(e^{2\pi it}, 1)}(A)dt.$$

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Exit time asymptotics on the non-commutative 2-torus

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Exit time asymptotics for non-commutative 2-torus

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Theorem

Let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *be a projection such that* f_0, f_1 *satisfy the condtions described before. Consider the projections* $A_{s,t}(P)$, $A_{s',t'}(P)$ *such that* $|s - s'| < \frac{\epsilon}{4}$. *Then*

$$(A_{s,t}(P)) \bigwedge \left(A_{s',t'}(P)\right) = \chi_S(U),$$

for the set $S = X_1 \cap X_2 \cap X_3 \cap X_4$, *where*
$X_1 = \tau_{-s}(\{x|f_1(x) = 0\}), X_2 := \tau_{-s'}(\{x|f_1(x) = 0\}),$
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It is worthwhile to note that the conclusion of the above theorem holds if we replace U by U^k, V by V^k, and θ by $\{k\theta\}$ ($\{\cdot\}$ denoting the fractional part).

Let $P_n = f_{-1}^{(k_n)}(U^{k_n}) + f_0^{(k_n)}(U^{k_n}) + f_1^{(k_n)}(U^{k_n})U^{k_n}$, be projections such that $\{k_n\theta\} \to 0$. Put $\epsilon := \frac{\{k_n\theta\}}{2}$.

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Define $j_t : W^*(\mathbb{T}_\theta^2) \to W^*(\mathbb{T}_\theta^2) \otimes B(\Gamma(L^2(\mathbb{R}_+, \mathbb{C}^2)))$ by
$j_t(\cdot) := \alpha_{(e^{2\pi i W_t^{(1)}}, e^{2\pi i W_t^{(2)}})}(\cdot)$.

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Note that j_t defined above is the standard Brownian motion on $C^*(\mathbb{T}_\theta^2)$.

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Almost surely, $\bigwedge_{s\leq t}(j_s(P_n)(\omega)) \in W^*(U)$*, for all* n*, i.e.*

$$\bigwedge_{s\leq t}(j_s(P_n)) \in W^*(U) \otimes B(\Gamma(L^2(\mathbb{R}_+,\mathbb{C}^2))),$$

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Outline of the proof:

In the strong operator topology,

$$\bigwedge_{0 \leq s \leq t} (j_s(P_n)) = \lim_{m \to \infty} \bigwedge_i \{ j_{\frac{it}{2^m}}(P_n) \wedge j_{\frac{(i+1)t}{2^m}}(P_n) \}. \tag{6}$$

Now almost surely a Brownian path restricted to $[0, t]$ is uniformly continuous, so that the for sufficiently large m, and for almost all ω, $|W^{(1)}_{\frac{it}{2^m}} - W^{(1)}_{\frac{(i+1)t}{2^m}}|$ can be made small, uniformly for all i such that $i = 0, 1, ..2^m$. So $\bigwedge_i \{ j_{\frac{it}{2^m}}(P_n) \wedge j_{\frac{(i+1)t}{2^m}}(P_n) \} \in W^*(U)$ by Theorem 3.2. It can be shown that the set of projections of this type is closed in the WOT-topology. Hence proved.

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Consider

$$\langle e(0), (\phi_{z_n} \otimes 1) \circ \bigwedge_{0 \le s \le t} (j_s(P_n)) e(0) \rangle.$$

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A direct computation shows that this is equal to

$$\mathbb{P}\{e^{2\pi i W_s^{(1)}} \in \mathcal{B},\ 0 \le s \le t\} = \mathbb{P}\{\tau_{[\frac{-\{k_n\theta\}}{4}, \frac{\{k_n\theta\}}{4}]} > t\},$$

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Consider the pure states $\{ev_z \circ E_1, ev_x \circ E_2 | x, z \in \mathbb{T}\}$ on $W^*(\mathbb{T}^2_\theta)$, which are also normal. Let $z_n = e^{2\pi i \frac{3\{k_n\theta\}}{4}}$. Consider the sequence of pure states $\phi_{z_n} := ev_{z_n} \circ E_1$.
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where $\mathcal{B} := \{e^{2\pi i x} : \ x \in [\frac{-\{k_n\theta\}}{4}, \frac{\{k_n\theta\}}{4}]\}$.
So we have a family of $(\tau_n)_n$ random times defined by

$$\tau_n([t, +\infty)) = \bigwedge_{0 \le s \le t} (j_s(P_n));$$

so that $\int_0^t \langle e(0), (\phi_{z_n} \otimes 1) \circ \bigwedge_{0 \le s \le t} (j_s(P_n)) e(0) \rangle dt$ can be taken as the expectation of the random time τ_n.

Note that here the analogue for balls of decreasing volume is $(P_n)_n$, such that $tr(P_n) = \{k_n\theta\} \to 0$, tr being the canonical trace in $W^*(\mathbb{T}^2_\theta)$.

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Now, by the Pinsky's result, we have

$$\begin{aligned}
&\int_0^t \langle e(0), (\phi_{z_n} \otimes 1) \circ \bigwedge_{0 \le s \le t} (j_s(P_n))e(0)\rangle dt \\
&= \mathbb{E}(\tau_{[\frac{-\{k_n\theta\}}{4}, \frac{\{k_n\theta\}}{4}]}) \\
&= 2\sin^2\left(\frac{\{k_n\theta\}}{8}\right) + \frac{2}{3}\sin^4\left(\frac{\{k_n\theta\}}{8}\right) + O\left(\sin^5\left(\frac{\{k_n\theta\}}{8}\right)\right) \\
&= \frac{\{k_n\theta\}^2}{2^5} + \frac{\{k_n\theta\}^4}{2^{11}.3} + O(\{k_n\theta\}^5),
\end{aligned} \tag{7}$$

since the mean curvature of the circle viewed inside $\mathbb{R}^2$ is 1.

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All these give a good justification for developing a general theory of quantum stochastic geometry.

THANK

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Let $\mathfrak{X} = \{A \in W^*(\mathbb{T}^2_\theta) |\ A = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V,\ f_1, f_0 \in L^\infty(\mathbb{T}), f_{-1}(t) := \overline{f_1(t+\theta)}\}$.

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Lemma

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Proof.

Let $A_\beta := f^{(\beta)}_{-1}(U)V^{-1} + f^{(\beta)}_0(U) + f^{(\beta)}_1(U)V$ be a convergent net in the ultraweak topology. Now $\phi_1(A_\beta) = f^{(\beta)}_0(U)$, $\phi_1(A_\beta V) = f^{(\beta)}_{-1}(U)$ and $\phi_1(A_\beta V^{-1}) = f^{(\beta)}_1(U)$ Since ϕ_1 is a normal map, which implies that $f^{(\beta)}_0(U)$, $f^{(\beta)}_1(U)$ *and* $f^{(\beta)}_{-1}(U)$ (all of which are elements of $L^\infty(\mathbb{T})$) are ultraweakly convergent, to $f_0(U), f_1(U), f_{-1}(U)$ (say), and clearly $f_{-1}(t) = \overline{f_1(t+\theta)}$. □

Exit time asymptotics on non-commutative 2-torus

Lemma

Suppose f_1, f_0 are as defined before and $A \in \mathfrak{X}$. Define

$$A_{s,t} := f_{-1}(e^{2\pi is}U)V^{-1}e^{-2\pi it} + f_0(e^{2\pi is}U) + f_1(e^{2\pi is}U)Ve^{2\pi it}.$$

Suppose $s, s' \in [0,1)$ be such that $|s-s'| \leq \frac{\epsilon}{4}$ where $0 < \epsilon < \theta$, and $|supp(f_1)| < \epsilon$, where $|C|$ denotes the Lebesgue measure of a Borel subset $C \subseteq \mathbb{R}$. Then $A_{s,t} \cdot A_{s',t'} \in \mathfrak{X}$.

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Proof.

It suffices to show that the coefficient of V^2 in $A_{s,t} \cdot A_{s',t'}$ is zero. By a direct computation, the coefficient of V^2 is $g(l) := f_1(s + l)f_1(s' + l - \theta)e^{2\pi i(t+t')}$. But $|(s + l) - (s' + l - \theta)| = |\theta + s - s'| > \epsilon$. Now by hypothesis, we have $|supp(f_1)| < \epsilon$, so that $f_1(s + l) \cdot f_1(s' + l - \theta) = 0$ and hence the lemma is proved. □

Exit time asymptotics on non-commutative 2-torus

Lemma

Suppose $A = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *and* $f_1(I)f_1(I+\theta) = 0$, *for* $I \in [0,1)$. *Then* $A^{2n} \in \mathfrak{X}$, *for* $n \in \mathbb{N}$.

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Proof.

The coefficient of V^2 in A^2 is $f_1(I)f_1(I+\theta)$ for $I \in [0,1)$ and this is zero by the hypoethesis. Hence $A^2 \in \mathfrak{X}$. The coefficient of V in A^2 is $f_1^{(2)}(I) := f_1\left(f_0 + \tau_\theta(f_0)\right)$, where τ_θ is left translation by θ. We have $f_1^{(2)}(I)f_1^{(2)}(I+\theta) = 0$, so that applying the same argument as before, we conclude that $A^4 \in \mathfrak{X}$. Proceeding like this we get the required result. □

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Exit time asympotics for the non-commutative 2-torus.

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Let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *and* $A = f_{-1}^{(A)}(U)V^{-1} + f_0^{(A)}(U) + f_1^{(A)}(U)V$ *be projections,* (f_{-1}, f_0, f_1) *and* $(f_{-1}^{(A)}, f_0^{(A)}, f_1^{(A)})$ *satisfying the conditions described before. Then* $A \leq A_{s,t}(P)$ *and* $A \leq A_{s',t'}(P)$ *if and only if the following hold:*
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3. $f_0(s+l)f_0^{(A)}(l) + f_1(s+l)f_{-1}^{(A)}(l-\theta)e^{2\pi it} + f_{-1}(s+l)f_1^{(A)}(l+\theta)e^{-2\pi it} = f_0^{(A)}(l);$

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1. $f_1(s+l)f_1^{(A)}(l-\theta) = 0;$
2. $f_{-1}(s+l)f_{-1}^{(A)}(l+\theta) = 0;$
3. $f_0(s+l)f_0^{(A)}(l) + f_1(s+l)f_{-1}^{(A)}(l-\theta)e^{2\pi it} + f_{-1}(s+l)f_1^{(A)}(l+\theta)e^{-2\pi it} = f_0^{(A)}(l);$
4. $f_1(s+l)f_0^{(A)}(l-\theta)e^{2\pi it} + f_0(s+l)f_1^{(A)}(l) = f_1^{(A)}(l);$

Exit time asympotics for the non-commutative 2-torus.

Lemma

Let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *and*
$A = f^{(A)}_{-1}(U)V^{-1} + f^{(A)}_0(U) + f^{(A)}_1(U)V$ *be projections,* (f_{-1}, f_0, f_1) *and* $(f^{(A)}_{-1}, f^{(A)}_0, f^{(A)}_1)$ *satisfying the conditions described before. Then* $A \leq A_{s,t}(P)$ *and* $A \leq A_{s',t'}(P)$ *if and only if the following hold:*
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4. $f_1(s+l)f^{(A)}_0(l-\theta)e^{2\pi it} + f_0(s+l)f^{(A)}_1(l) = f^{(A)}_1(l);$
5. $f_{-1}(s+l)f^{(A)}_0(l+\theta)e^{-2\pi it} + f_0(s+l)f^{(A)}_{-1}(l) = f^{(A)}_{-1}(l);$

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Let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *and* $A = f^{(A)}_{-1}(U)V^{-1} + f^{(A)}_0(U) + f^{(A)}_1(U)V$ *be projections,* (f_{-1}, f_0, f_1) *and* $(f^{(A)}_{-1}, f^{(A)}_0, f^{(A)}_1)$ *satisfying the conditions described before. Then* $A \leq A_{s,t}(P)$ *and* $A \leq A_{s',t'}(P)$ *if and only if the following hold:*
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5. $f_{-1}(s + l)f^{(A)}_0(l + \theta)e^{-2\pi it} + f_0(s + l)f^{(A)}_{-1}(l) = f^{(A)}_{-1}(l);$
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Exit time asympotics for the non-commutative 2-torus.

Lemma

Let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *and* $A = f_{-1}^{(A)}(U)V^{-1} + f_0^{(A)}(U) + f_1^{(A)}(U)V$ *be projections,* (f_{-1}, f_0, f_1) *and* $(f_{-1}^{(A)}, f_0^{(A)}, f_1^{(A)})$ *satisfying the conditions described before. Then* $A \le A_{s,t}(P)$ *and* $A \le A_{s',t'}(P)$ *if and only if the following hold:*
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4. $f_1(s+l)f_0^{(A)}(l-\theta)e^{2\pi it} + f_0(s+l)f_1^{(A)}(l) = f_1^{(A)}(l);$
5. $f_{-1}(s+l)f_0^{(A)}(l+\theta)e^{-2\pi it} + f_0(s+l)f_{-1}^{(A)}(l) = f_{-1}^{(A)}(l);$
6. $f_1(s'+l)f_1^{(A)}(l-\theta) = 0;$
7. $f_{-1}(s'+l)f_{-1}^{(A)}(l+\theta) = 0;$

Exit time asympotics for the non-commutative 2-torus.

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Let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *and* $A = f^{(A)}_{-1}(U)V^{-1} + f^{(A)}_0(U) + f^{(A)}_1(U)V$ *be projections,* (f_{-1}, f_0, f_1) *and* $(f^{(A)}_{-1}, f^{(A)}_0, f^{(A)}_1)$ *satisfying the conditions described before. Then* $A \leq A_{s,t}(P)$ *and* $A \leq A_{s',t'}(P)$ *if and only if the following hold:*
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4. $f_1(s+l)f^{(A)}_0(l-\theta)e^{2\pi it} + f_0(s+l)f^{(A)}_1(l) = f^{(A)}_1(l);$
5. $f_{-1}(s+l)f^{(A)}_0(l+\theta)e^{-2\pi it} + f_0(s+l)f^{(A)}_{-1}(l) = f^{(A)}_{-1}(l);$
6. $f_1(s'+l)f^{(A)}_1(l-\theta) = 0;$
7. $f_{-1}(s'+l)f^{(A)}_{-1}(l+\theta) = 0;$
8. $f_0(s'+l)f^{(A)}_0(l)+f_1(s'+l)f^{(A)}_{-1}(l-\theta)e^{2\pi it'}+f_{-1}(s'+l)f^{(A)}_1(l+\theta)e^{-2\pi it'} = f^{(A)}_0(l);$

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4. $f_1(s + l)f^{(A)}_0(l - \theta)e^{2\pi it} + f_0(s + l)f^{(A)}_1(l) = f^{(A)}_1(l);$
5. $f_{-1}(s + l)f^{(A)}_0(l + \theta)e^{-2\pi it} + f_0(s + l)f^{(A)}_{-1}(l) = f^{(A)}_{-1}(l);$
6. $f_1(s' + l)f^{(A)}_1(l - \theta) = 0;$
7. $f_{-1}(s' + l)f^{(A)}_{-1}(l + \theta) = 0;$
8. $f_0(s'+l)f^{(A)}_0(l)+f_1(s'+l)f^{(A)}_{-1}(l-\theta)e^{2\pi it'}+f_{-1}(s'+l)f^{(A)}_1(l+\theta)e^{-2\pi it'} = f^{(A)}_0(l);$
9. $f_1(s' + l)f^{(A)}_0(l - \theta)e^{2\pi it'} + f_0(s' + l)f^{(A)}_1(l) = f^{(A)}_1(l);$

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Let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ *and* $A = f_{-1}^{(A)}(U)V^{-1} + f_0^{(A)}(U) + f_1^{(A)}(U)V$ *be projections,* (f_{-1}, f_0, f_1) *and* $(f_{-1}^{(A)}, f_0^{(A)}, f_1^{(A)})$ *satisfying the conditions described before. Then* $A \leq A_{s,t}(P)$ *and* $A \leq A_{s',t'}(P)$ *if and only if the following hold:*

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3. $f_0(s+l)f_0^{(A)}(l)+f_1(s+l)f_{-1}^{(A)}(l-\theta)e^{2\pi it}+f_{-1}(s+l)f_1^{(A)}(l+\theta)e^{-2\pi it} = f_0^{(A)}(l);$
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5. $f_{-1}(s+l)f_0^{(A)}(l+\theta)e^{-2\pi it} + f_0(s+l)f_{-1}^{(A)}(l) = f_{-1}^{(A)}(l);$
6. $f_1(s'+l)f_1^{(A)}(l-\theta) = 0;$
7. $f_{-1}(s'+l)f_{-1}^{(A)}(l+\theta) = 0;$
8. $f_0(s'+l)f_0^{(A)}(l)+f_1(s'+l)f_{-1}^{(A)}(l-\theta)e^{2\pi it'}+f_{-1}(s'+l)f_1^{(A)}(l+\theta)e^{-2\pi it'} = f_0^{(A)}(l);$
9. $f_1(s'+l)f_0^{(A)}(l-\theta)e^{2\pi it'} + f_0(s'+l)f_1^{(A)}(l) = f_1^{(A)}(l);$
10. $f_{-1}(s'+l)f_0^{(A)}(l+\theta)e^{-2\pi it'} + f_0(s'+l)f_{-1}^{(A)}(l) = f_{-1}^{(A)}(l);$

Exit time asymptotics for non-commutative 2-torus

Lemma

For two projections A and B such that

$$A = f_{-1}^{(A)}(U)V^{-1} + f_0^{(A)}(U) + f_1^{(A)}(U)V,$$

$$B = f_{-1}^{(B)}(U)V^{-1} + f_0^{(B)}(U) + f_1^{(B)}(U)V;$$

we have $A \leq B$ if and only if for $l \in [0,1)$, we have:

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For two projections A and B such that

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we have $A \leq B$ *if and only if for* $l \in [0,1)$*, we have:*

- $f_1^{(B)}(l)f_1^{(A)}(l-\theta) = 0;$

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$$B = f_{-1}^{(B)}(U)V^{-1} + f_0^{(B)}(U) + f_1^{(B)}(U)V;$$

we have $A \leq B$ *if and only if for* $I \in [0,1)$, *we have:*

- $f_1^{(B)}(I)f_1^{(A)}(I-\theta) = 0;$
- $f_1^{(B)}(I+\theta)f_1^{(A)}(I+2\theta) = 0;$

Exit time asymptotics for non-commutative 2-torus

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For two projections A and B such that

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$$B = f_{-1}^{(B)}(U)V^{-1} + f_0^{(B)}(U) + f_1^{(B)}(U)V;$$

we have $A \leq B$ *if and only if for* $I \in [0, 1)$, *we have:*

- $f_1^{(B)}(I)f_1^{(A)}(I - \theta) = 0;$
- $f_1^{(B)}(I + \theta)f_1^{(A)}(I + 2\theta) = 0;$
- $f_0^{(B)}(I)f_0^{A}(I) + f_1^{(B)}(I)f_0^{(A)}(I) + f_1^{(B)}(I + \theta)f_1^{(A)}(I + \theta) = f_0^{(A)}(I);$

Lemma

For two projections A and B such that

$$A = f_{-1}^{(A)}(U)V^{-1} + f_0^{(A)}(U) + f_1^{(A)}(U)V,$$

$$B = f_{-1}^{(B)}(U)V^{-1} + f_0^{(B)}(U) + f_1^{(B)}(U)V;$$

we have $A \leq B$ if and only if for $l \in [0,1)$, we have:

- $f_1^{(B)}(l)f_1^{(A)}(l-\theta) = 0;$
- $f_1^{(B)}(l+\theta)f_1^{(A)}(l+2\theta) = 0;$
- $f_0^{(B)}(l)f_0^{A}(l) + f_1^{(B)}(l)f_0^{(A)}(l) + f_1^{(B)}(l+\theta)f_1^{(A)}(l+\theta) = f_0^{(A)}(l);$
- $f_1^{(B)}(l)f_0^{(A)}(l-\theta) + f_0^{(B)}(l)f_1^{(A)}(l) = f_1^{(A)}(l);$

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For two projections A and B such that

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we have $A \leq B$ *if and only if for* $l \in [0,1)$, *we have:*

- $f_1^{(B)}(l)f_1^{(A)}(l-\theta) = 0;$
- $f_1^{(B)}(l+\theta)f_1^{(A)}(l+2\theta) = 0;$
- $f_0^{(B)}(l)f_0^{A}(l) + f_1^{(B)}(l)f_0^{(A)}(l) + f_1^{(B)}(l+\theta)f_1^{(A)}(l+\theta) = f_0^{(A)}(l);$
- $f_1^{(B)}(l)f_0^{(A)}(l-\theta) + f_0^{(B)}(l)f_1^{(A)}(l) = f_1^{(A)}(l);$
- $f_1^{(B)}(l+\theta)f_0^{(A)}(l+\theta) + f_0^{(B)}(l)f_1^{(A)}(l+\theta) = f_1^{(A)}(l+\theta);$

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For two projections A and B such that

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we have $A \leq B$ *if and only if for* $l \in [0,1)$, *we have:*

- $f_1^{(B)}(l)f_1^{(A)}(l-\theta) = 0;$
- $f_1^{(B)}(l+\theta)f_1^{(A)}(l+2\theta) = 0;$
- $f_0^{(B)}(l)f_0^{A}(l) + f_1^{(B)}(l)f_0^{(A)}(l) + f_1^{(B)}(l+\theta)f_1^{(A)}(l+\theta) = f_0^{(A)}(l);$
- $f_1^{(B)}(l)f_0^{(A)}(l-\theta) + f_0^{(B)}(l)f_1^{(A)}(l) = f_1^{(A)}(l);$
- $f_1^{(B)}(l+\theta)f_0^{(A)}(l+\theta) + f_0^{(B)}(l)f_1^{(A)}(l+\theta) = f_1^{(A)}(l+\theta);$

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we have $A \leq B$ if and only if for $l \in [0,1)$, we have:

- $f_1^{(B)}(l)f_1^{(A)}(l-\theta) = 0;$
- $f_1^{(B)}(l+\theta)f_1^{(A)}(l+2\theta) = 0;$
- $f_0^{(B)}(l)f_0^{A}(l) + f_1^{(B)}(l)f_0^{(A)}(l) + f_1^{(B)}(l+\theta)f_1^{(A)}(l+\theta) = f_0^{(A)}(l);$
- $f_1^{(B)}(l)f_0^{(A)}(l-\theta) + f_0^{(B)}(l)f_1^{(A)}(l) = f_1^{(A)}(l);$
- $f_1^{(B)}(l+\theta)f_0^{(A)}(l+\theta) + f_0^{(B)}(l)f_1^{(A)}(l+\theta) = f_1^{(A)}(l+\theta);$

Lemma

Let $P = f_{-1}(U)V^{-1} + f_0(U) + f_1(U)V$ such that P is a projection and suppose $f_0(t) = 0$ for some t. Then $f_1(t) = f_1(t+\theta) = 0$.